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## Bernard Bolzano's Contributions to Logic and Ontology. Second Part: C - Geo

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## Bernard Bolzano. Annotated Bibliography on His Practical Philosophy

### Studies on Bolzano's Logic and Ontology

1. Cantù, Paola. 2011. "Bolzano Versus Kant: Mathematics as a *Scientia Universalis*." In *Mind, Values, and Metaphysics: Philosophical Essays in Honor of Kevin Mulligan*. Vol. 1, edited by Reboul, Anne. Dordrecht: Springer  
 Abstract: "The chapter will discuss some changes in Bolzano's definition of mathematics attested in several quotations from the *Beyträge*, *Wissenschaftslehre* and *Größenlehre*: Is mathematics a theory of forms or a theory of quantities? Several issues that are maintained throughout Bolzano's works will be distinguished from others that were accepted in the *Beyträge* and abandoned in the *Größenlehre*. Changes will be interpreted not only as a consequence of the new logical theory of truth introduced in the *Wissenschaftslehre* but also as a consequence of the overcome of Kant's terminology, and of the radicalization of Bolzano's anti-Kantianism.  
 It will be argued that Bolzano's evolution can be understood as a coherent move, if one compares the criticism on the notion of quantity expressed in the *Beyträge* with a different and larger notion of quantity that Bolzano developed already in 1816. This discussion is based on the discovery that two unknown texts mentioned by Bolzano can be identified with works by von Spaun and Vieth. Bolzano's evolution will be interpreted as a radicalization of the criticism of the Kantian definition of mathematics and as an effect of Bolzano's unaltered interest in the Leibnizian notion of *mathesis universalis*. As a conclusion, it will be argued that Bolzano never abandoned his original idea of considering mathematics as a *scientia universalis*, i.e. as the science of quantities in general, and it will be suggested that the question of ideal elements in mathematics, which has been interpreted as a main reason for the development of a new logical theory, can also be considered as a main reason for developing a different definition of quantity."
- References
- Vieth G.U.A. (1805) *Anfangsgründe der Mathematik. Lehrbuch der angewandten Elementarmathematik*. Barth, Leipzig-
- von Spaun F.A.R. (1805) *Versuch das Studium der Mathematik durch Erläuterung einiger Grundbegriffe und durch zweckmässigere Methoden zu erleichtern*. Göbhardt, Bamberg-
2. Casari, Ettore. 1989. "Remarks on Bolzano's Modalities." In *Atti del Convegno Internazionale di Storia della Logica: le teorie della modalità*, edited by Corsi, Giovanna, Mangione, Corrado and Mugnai, Massimo, 319-322. Bologna: CLUEB "1. The Roots of Bolzano's Interest in Modalities.  
 From an autobiographical note quoted by Winter ([1], p.32), we learn that as Bolzano was 17 years old and began to read the *Kritik der reinen Vernunft*, he was soon strongly attracted by the distinctions of Judgments into a priori and a posteriori and into analytic and syntetic as well as by the distinction of representations into Intuitions and concepts, whereas he was very hurted by the immediate use, without any previous explanation, of the concepts of experience and of necessity. A significant part of Bolzano's work may be seen as an attempt to clarify the preceding notions and to substantiate their distinctions. In particular, his theory of (absolute) modalities is his answer to the question about necessity, an answer which he derives from his answer to the question about experience. Necessity is indeed first reduced to necessary truth; this latter is identified with true proposition in Itself which doesn't depend upon experience; depending upon experience is identified with containing intuitions; intuitions are a logically well defined kind of representations In themselves (Ideas).

Bolzano's theory of modalities has been scarcely considered up to now; the most careful analysis has been done by E. Morscher ([2], pp.87-92). In the following we will embed Bolzano's theory of the absolute modalities, as presented mainly in the *Wissenschaftslehre* §182, into the general framework of his logic we have reconstructed elsewhere ([3], [4]) and which will be only sketched here. We will not consider his theory of relative modalities." (p. 319)

(...)

"*Last Remarks.* As alluded to in §1, Bolzano's primary interest was in the notion of necessary truth. The whole of his modal theory, although interesting in many respects, is far from being satisfactorily refined. So, for instance, we remark that according to Bolzano, every standard proposition whose subject is unobjectual [*gegenstandlos*], that is, referring to no object, is false, it follows that all such propositions are possible (although their being true may be Impossible). That truth is a quality and that there is an Idea which refers to it, are, of course, rather disquieting assumptions. From their discussion in [4], it follows that a very important question is whether the idea [p], which refers to p, has also p as its part. In the present context, the question presents itself, in particular, with respect to the problem about the conceptuality of a proposition having the idea [p] as its subject, under the hypothesis of the conceptuality of the proposition p and vice versa; a problem which immediately arises, when considering, for instance, iterated modalities." (p. 323)

#### Bibliographical Note

- [1] E. Winter, *Die geistige Entwicklung Bolzanos*, in E. Winter, P. Funk, J. Berg, *Bernard Bolzano, Ein Denker und Erzieher Im Österreichischen Vormärz*, Sitz.-Ber. d. Öst. Ak.d. Wiss., Phil.-Hist.Kl., Bd. 252, Abhdl. 5, Hf. 8, Wien 1967, pp. 29-74.
- [2] E. Morscher, *Philosophische Logik bei Bernard Bolzano*, in Bolzano-Symposion: "Bolzano als Logiker", Sitz.-Ber. d. Öst. Ak.d. Wiss., Phil.-Hist. Kl., Bd.293, Abhdl. 5, Hf. 12, Wien 1974, pp. 77-105.
- [3] E. Casari, *Bemerkungen über die Bolzanosche Wissenschaftslehre*, in *Logik und Grundlagenforschung*, H.Scholz-Kolloquium, Aschendorff, Münster i. W., 1985, pp. 53-66.
- [4] E. Casari, *An Interpretation of some ontological and semantical notions In Bolzano's logic*, to appear in the proc. of the meet. (1987) of the Florence Center for Hist. and Phil. of Sc.: *Wissenschaftslehre 1B37-1987*.
- 3. ———. 1992. "An Interpretation of Some Ontological and Semantical Notions in Bolzano's Logic." In *Bolzano's Wissenschaftslehre 1837-1987. International Workshop*, 55-105. Firenze: Leo S. Olschki  
"In the following, the attempt is done to clarify some significant features of Bolzano's logical system with particular attention to its development in the *Wissenschaftslehre* (WL). This system is viewed as a theory trying to identify certain quite general properties, relations and operations of *things* [Dinge], in the most general and unbiased sense of this word. For sake of simplicity and determinateness of the formulations, the current logical symbolism is *used*. The point of view of the theory is *elementary*, that is to say, we always work with *particular notions* about the things without allowing us any consideration of *arbitrary notions* about the things. 'x', 'y', 'z', ... are used as variables for things." (p. 55)
- 4. ———. 2006. "Some Remarks on Bolzano's Notion of a Quality." In *Logic and Philosophy in Italy. Some Trends and Perspectives. Essays in honor of Corrado Mangione*, edited by Ballo, Edoardo and Franchella, Miriam, 185-201. Milano: Polimetrica.
- 5. ———. 2016. *Bolzano's Logical System*. Oxford: Oxford University Press  
"As already mentioned, many specialist studies have analysed many of the questions that arise from the first three parts of Bolzano's work, as well as providing comprehensive expositions of them, often very successfully. Yet, it seems to us that

there remains room for a more systematic reconsideration of Bolzano's logical thought.

This book is concerned precisely with this aim. In undertaking this task, the book is intended as an exploration, not so much of the more specifically discursive aspects of Bolzano's logical thought—already amply studied—as much as one aimed at identifying the singularly coherent and systematic nature of the logic presented in the *Wissenschaftslehre*.

In order to render as visible as possible the systematic nature of that logic, I have decided to present it within a formal system. Despite being surprising even to me, it has become clear that in pursuing this aim, it is sufficient to adopt the approach of the predicate calculus with identity and choice operator, that is, enlisting the wellknown Hilbert's epsilon calculus. As this book reveals, the formalization of Bolzano's logic in this calculus emerges quite effortlessly." (Preface, p. VIII)

6. ———. 2017. "Husserl and Bolzano." In *Essays on Husserl's Logic and Philosophy of Mathematics*, edited by Centrone, Stefania, 75-91. Springer  
Abstract: "The paper examines the all too often neglected role of the Czech philosopher and mathematician Bernard Bolzano for Husserl's work, from ca. 1893–1894 onwards. Husserl himself finds it important to stress in an appendix to chapter 10 of the *Prolegomena to Pure Logic* that his investigations are not "in any sense mere commentaries upon, or critically improved expositions of, Bolzano's thought patterns", but that they "have been crucially stimulated by Bolzano... ". The paper examines early Bolzano's ideas on the ground-consequence relation, Bolzano's logical universe as presented in his masterpiece, the monumental *Wissenschaftslehre*, the role of Hermann Lotze in making Husserl receptive for Bolzano and, finally, a lecture course on logic held by Husserl at the University of Halle in 1896 [\*], working out just what Husserl is taking, and not taking, from Bolzano."  
[\*] E. Husserl, [LV'96] *Logik: Vorlesung 1896*, ed. by E. Schuhmann. *Husserliana Materialienbände I* (Kluwer, Dordrecht, 2001)
7. Cellucci, Carlo. 1992. "Bolzano and Multiple-Conclusion Logic." In *Bolzano's Wissenschaftslehre 1837-1987. International Workshop*, 179-189. Firenze: Leo S. Olschki  
"The aim of this paper is to assess Bolzano's logical work in the light of contemporary logical developments. This has been done before by others, most recently by van Benthem, (1) but everybody has his own approach and my approach -- whatever its value -- will be somewhat different from the current one. Make no mistake, I am not going to discuss once again to what extent Bolzano anticipated modern logic. On the contrary I will try to show how far he was from moden logic. In order to do so I will compare Bolzano with the tradition of multiple-conclusion logic." (p. 179)  
(1) J. van Benthem, *The Variety of Consequence, According to Bolzano*, Studia Logica 44, 1985, pp. 389-403.
8. Centrone, Stefania. 2010. "Functions in Frege, Bolzano and Husserl." *History and Philosophy of Logic* no. 31:315-336  
Abstract: "This explorative article is organized around a set of questions concerning the concept of a function. First, a summary of certain general facts about functions that are a common coin in contemporary logic is given. Then Frege's attempt at clarifying the nature of functions in his famous paper *Function and Concept* and in his *Grundgesetze* is discussed along with some questions which Frege's approach gave rise to in the literature. Finally, some characteristic uses of functional notions to be found in the work of Bernard Bolzano and in Edmund Husserl's early work are presented and elucidated."  
"4. Bernard Bolzano  
In this section, I want to show that the set-theoretical notion of a function is implicitly at work in Bolzano's logic of variation. Bolzano's own use of the term 'function' is not pertinent here, for he employs this term only in the context of 'x is

a function of y1, y2,..., yn' where the correlated entities are what he calls *Größen* (magnitudes).(24) Thus, Bolzano's usage of the term (unlike Frege's) is restricted to the field of mathematics. The entities his logic of variation is concerned with are not magnitudes, but propositions and their non-propositional parts." (p. 325)  
 (24) Bolzano 1830–1835, [J. Berg, ed., *Einleitung zur Größenlehre. Erste Begriffe der allgemeinen Größenlehre*, BGA Series 2A, vol. 7, 1975], p. 229.

9. ———. 2016. "Early Bolzano on *Ground-Consequence Proofs*." *The Bulletin of Symbolic Logic* no. 2:215-237  
 Abstract: "In his early *Contributions to a Better-Grounded Presentation of Mathematics* (1810) Bernard Bolzano tries to characterize rigorous proofs (*strenge Beweise*). Rigorous is, *prima facie*, any proof that indicates the grounds for its conclusion. Bolzano lists a number of methodological constraints all rigorous proofs should comply with, and tests them systematically against a specific collection of elementary inference schemata that, according to him, are evidently of ground-consequence-kind. This paper intends to give a detailed and critical account of the fragmentary logic of the *Contributions*, and to point out as well some difficulties Bolzano's attempt runs into, notably as to his methodological ban on 'kind crossing'."
10. Chattopadhyaya, Debi Prasad. 1979. "Bolzano and Frege: A Note on Ontology." In *Logic, Ontology and Action*, edited by Banerjee, K.K., 214-242. Atlantic Highlands: Humanities Press.
11. Chihara, Charles. 1999. "Frege's and Bolzano's Rationalist Conceptions of Arithmetic." *Revue d'Histoire des Sciences* no. 52:343-361  
 Abstract: "In this article, I compare Gottlob Frege's and Bernard Bolzano's rationalist conceptions of arithmetic. Each philosopher worked out a complicated system of propositions, all of which were set forth as true. The axioms, or basic truths, make up the foundations of the subject of arithmetic. Each member of the system which is not an axiom is related (objectively) to the axioms at the base. Even though this relation to the base may not yet be scientifically proven, the propositions of the system include all of the truths of the science of arithmetic. I conclude the article by analyzing the respective views of Frege and Bolzano in the light of Gödel's first incompleteness theorem."
12. Chisholm, Roderick M. 1986. "On the Positive and Negative State of Things." In *Non-Existence and Predication*, edited by Haller, Rudolf, 97-106. Amsterdam: Rodopi  
 Abstract: "Following Balzano, I suggest that there are two types of entity: those that are states of other things and those that are not. The second type includes, not only substances, in the traditional sense, but also such abstract objects as numbers, attributes and propositions. It is argued that the theory of states, when combined with an intentional account of negative attributes, will yield a theory of negative entities and of events."
13. ———. 1986. "The Self in Austrian Philosophy." In *Von Bolzano zu Wittgenstein. Zur Tradition der österreichischen Philosophie = From Bolzano to Wittgenstein. The Tradition of Austrian Philosophy*, edited by Nyíri, János Kristóf 71-74. Wien: Hölder-Pichler-Tempsky  
 Reprinted in: R. M. Chisholm, *On Metaphysics*, Minneapolis: University of Minnesota Press, 1989, pp. 156-161.  
 "Bolzano's definition of substance provides us with a kind of key to the conceptions of the self in Austrian philosophy. His definition is as clear as anyone could possibly wish. He says that there are two kinds of things: (I) those things that are states or conditions of other things ("Beschaffenheiten von anderen Dingen"); and (II) those things that are not states or conditions of other things: "the latter are what I call *substances*."(1) Examples of things that are states or conditions of other things are "the color, smell and weight of a body," the beliefs that a particular person has, the sensations that he has, and the actions that he performs. Examples of substances-of

things that are not states or conditions of other things -are physical bodies and selves.

Bolzano says, in Leibnizian fashion, that, if there are things that are states or conditions of other things, then there are things that are not states or conditions of other things.(2) If we use the term "substance" in the way he suggests, then we need not ask whether a given philosopher believes in substances; we need ask only what the things are that function for him as substances." (p. 156 of the reprint)

(1) Bolzano, *Athanasia oder Gründe für die Unsterblichkeit der Seele* (Sulzbach: J. G. v. Seidleschen Buchhandlung, 1838), [second enlarged edition; first edition 1827] p. 283.

(2) Bolzano (1827), p. 22. He holds that *Beschaffenheiten* may themselves have *Beschaffenheiten* and that such things as numbers also have *Beschaffenheiten* (p. 22), and he seems to hold that God has a *Beschaffenheit* (p. 22).

14. ———. 1989. "Bolzano on the Simplicity of the Soul." In *Traditionen und Perspektiven der analytischen Philosophie*, edited by Gombocz, Wolfgang, Rute, Heiner and Sauer, Werner. Vienna: Hölder-Pichler-Tempsky.
15. ———. 1991. "Bernard Bolzano's Philosophy of Mind." *Philosophical Topics* no. 19:207-216  
 "The views of Bernard Bolzano ( 1781-1842) concerning the nature of psychological properties and the nature of what it is that has those properties are of first importance to philosophy. I shall discuss some of them here in the hope that what I say may lead to a more systematic study and evaluation.  
 Bolzano's best known works are the *Theory of Science* [Wissenschaftslehre], first published in 1837, and *The Paradoxes of the Infinite* [Paradoxien des Unendlichen], first published in 1851. The present topic is discussed in detail in *Athanasia: Or Grounds for the Immortality of the Soul*, published in 1838.(2) This work has not been translated into English." (p. 205, note 1 omitted)  
 (...)  
 "Bolzano, then, is concerned with presenting considerations which, he thinks, indicate that only simple substances can think. In order to avoid a fundamental misunderstanding, we must be clear about one fundamental point. In setting out on his investigations, Bolzano assumes that it is not known that thinking things are identical with physical bodies. Hence he does not presuppose the thesis according to which we are identical with our bodies or with some proper part of our bodies. Others, of course, may presuppose the contrary of this thesis. But any *criticism* of Bolzano that is based upon the contrary thesis and that does not include a positive defence of this contrary thesis would be question begging." (p. 207)  
 (2) *Athanasia; oder Gründe für die Unsterblichkeit der Seele* (Sulzbach: J. G. v. Seidleschen Buchhandlung, 1838). [Second enlarged editon; first edition 1827.]
16. Claas, Jan. 2021. "Leibniz and Bolzano on Conceptual Containment." *European Journal of Philosophy*:1-19  
 Abstract: "Philosophers often rely on the notion of conceptual containment and apply mereological terminology when they talk about the parts or constituents of a complex concept. In this paper, I explore two historical approaches to this general notion. In particular, I reconstruct objections Bernard Bolzano puts forward against a criterion that played a prominent role in the history of philosophy and that was endorsed, among others, by Leibniz. According to this criterion, a concept that represents objects contains all and only the concepts that represent properties the objects must have in order to be represented by the former concept.  
 Bolzano offers several counterexamples and arguments against the criterion. I argue that while some of them presuppose a strongly mereological understanding of containment, which Leibniz is not committed to, one of them also succeeds without relying on demanding mereological principles."
17. Coffa, J. Alberto. 1982. "Kant, Bolzano and the Emergence of Logicism." *Journal of Philosophy* no. 74:679-689

"Bolzano was the first to recognize the fallacy behind the principle of synthetic judgments. The crucial step in Kant's inference for the need to appeal to intuition in synthetic judgments was the premise that from concepts alone only analytic knowledge can be derived. Astonishingly, there isn't a single argument in the Critique for this claim; all Kant says about it is that "it is evident" (A47, B64).(6) What is evident, instead, is that Kant had confused true in virtue of concepts with true in virtue of definitions, or, in his own language, he had erroneously identified judgments whose predicate is not contained in their subject-concept with judgments that extend our knowledge (*Erweiterungsurteile*). Against this, Bolzano was the first to make a point that even Frege would miss: that Kant's analytic judgments, far from exhausting the grounding power of the conceptual resources of our language, mobilize only a very modest fraction of them, the logical concepts. Bolzano's characterization of analyticity is well known, and it has often been noted that it anticipates not Frege's proof-theoretic treatment but the more modern semantic approach by means of interpretations. What is less well known is the reasoning that led Bolzano to this proposal. After reviewing a number of attempts to explain the point of Kant's notion of analyticity, Bolzano comments that "none of these explanations singles out what makes these [analytic] propositions important. I believe that this consists in the fact that their truth or falsity does not depend upon their constituent representations but remains unaltered, whatever changes one may make in some of these representations . . . This is the ground of my preceding definition."(7) Thus, the reason why Bolzano came to his celebrated insight on the semantic characterization of logical truth is that he saw that Kant's analytic judgments, far from being those grounded on the information implicit in the constituent concepts, were grounded on only a few of those concepts, thus concluding that a proper definition of analyticity should emphasize the extent to which all other concepts are to be ignored." (p. 684)

(6) For a very modest effort toward an argument, see Ak 20, 340.

(7) *Wissenschaftslehre* (Hamburg: Meiner, 1929), vol. II, sec. 148, p. 88.

18. ———. 1991. *The Semantic Tradition from Kant to Carnap*. Cambridge: Cambridge University Press  
 Second Chapter: *Bolzano and the Birth of Semantics* pp. 22-40.  
 "While the idealists were removing every trace of objectivity from Kant's semantics, there was in a corner of the Austro-Hungarian empire, ignored by the leaders of German philosophy, a Czech priest by the name of Bernard Bolzano, who was engaged in the most far-reaching and successful effort to date to take semantics out of the swamp into which it had been sinking since the days of Descartes. Bolzano was the first to recognize that transcendental philosophy and its idealistic sequel were a *reductio ad absurdum* of the semantics of modern philosophy. He was also the first to see that the proper prolegomena to any future metaphysics was a study not of transcendental considerations but of what we say and its laws and that consequently the *prima philosophia* was not metaphysics or ontology but semantics. The development of these ideas in his monumental *Wissenschaftslehre* and in a variety of other writings established Bolzano as the founder of the semantic tradition.  
 Bolzano's philosophy was the kind that takes from and then gives life to science. His approach to semantics was developed in dialectical interplay with his decision to solve certain problems concerning the nature of mathematical knowledge. Kant had not even seen these problems; Bolzano solved them. And his solutions were made possible by, and were the source of, a new approach to the content and character of a priori knowledge. We shall illustrate the point by focusing on one of Bolzano's favorite mathematical topics, the calculus." (p. 23)
19. Cohen, Jonathan L. 1982. "Bolzano's Theory of Induction." In *Impact of Bolzano's Epoch on Development of Science - Conference Papers Prague 1981*, 443-457. Prague: Ustav československých a světových dějin CSAV  
 Also published in: Merrilee H. Salmon (ed.), *The Philosophy of Logical Mechanism*, Dordrecht: Springer 2011, pp. 29-40.

Abstract: "Bolzano's *Wissenschaftslehre* was published in 1837, although most of it seems to have been written during the decade 1820–1830. John Stuart Mill's *System of Logic* was published in 1843, but had been in gestation or preparation since 1825. Neither author seems to have exercised any influence on the other, and in their views about the fundamental nature of logical and mathematical reasoning they notoriously represented very different trends. Bolzano sought to direct philosophers' attention away from mental processes towards relationships between ideas in themselves and between propositions in themselves, while Mill's logic insisted on a study of the mental process which takes place whenever we reason, of the conditions on which this process depends, and of the steps of which it consists. But in their views about the methodology of natural science the divergences are much more finegrained. Both assign a central role to the search for causes and both discuss the same basic procedures for the discovery of these. It is just that Bolzano shows a greater sensitivity than Mill does to the inherent difficulties of the enterprise."

20. Corcoran, John. 1975. "Meanings of Implication." *Diálogos* no. 9:59-76  
 Reprinted in: R. I. G. Hughes (ed.), *A Philosophical Companion to First-order Logic*, Indianapolis: Hackett 1993, pp. 85-100.  
 "In philosophical and mathematical discourse as well as in ordinary scholarly contexts the term 'implies' is used in several clear senses, many of which have already been noticed and explicated. The first five sections of this article codify and interrelate the most widely recognized meanings. Section 6 discusses a further significant and common use. Section 7 discusses and interrelates Tarski's notion of logical consequence, the "model-theoretic" notion of logical consequence, and Bolzano's two grounding relations. The eighth section employs the use-mention distinction to separate the three common grammatical categories of 'implies'. Section 8 also shows that criteria based on use-mention are not reliable indications of intended usage of 'implies'. The ninth and last section relates the above to the counterfactual and gives reasons for not expecting to find 'implies' used to express counterfactuals. A summary is provided."  
 "Summary and Conclusion: In the first five sections we have distinguished twelve uses of the term 'implies'. At the outset we distinguished: implies1 (truth-functional), implies2 (logical consequence) and implies3 (logical deducibility). Next we distinguished three elliptical or enthymematic varieties of implication: C-implies1, C-implies2 and C-implies3. In none of these six senses did "A implies B" presuppose the truth of A. Then we discussed the cases wherein "A implies B" is used to mean "The-fact-that-A implies B," which does presuppose the truth of A. We paraphrased the latter as "A is true and A implies 13" where 'implies' indicates any of the previous six senses of the term. Thus, at that point, twelve senses of implies were distinguished, six which do not presuppose the truth of the implying sentence and six which do. Of the six which do, three are enthymematic.  
 In addition, the three original senses were carefully distinguished and interrelated, and possible causes of confusion were identified.  
 Then, building on some off-hand observations of Russell, we related the truth-functional use of 'implies' to two further notions which have been used as explications of traditional logical consequence. We also brought in Bolzano's relative implication and his two grounding relations.  
 We argued briefly that counterfactuals are not normally expressed using 'implies' and that the distinction between use and mention cannot be used as a test for distinguishing different meanings of 'implies'.  
 Use of 'implies' as a transitive verb taking a human subject has been ignored."  
 21. de Jong, Willem R. 2001. "Bernard Bolzano, Analyticity and the Aristotelian Model of Science." *Kant-Studien* no. 91:328-349  
 "In this article I intend to make clear that Bolzano's perception and use of the distinction in question [analytic-synthetic] should also be understood in the framework of this model of science. The effect of doing so is to render more comprehensible Bolzano's highly personal and, in its application, upon first

acquaintance rather strange characterization of the analytic-synthetic distinction. This characterization can then also be placed more easily in its historical context. [Joëlle] Proust aside, most interpreters have looked somewhat askance at Bolzano's notion of analyticity. And most of them seem not to be able to go on and do much with this apparently anomalous element in Bolzano's thinking. (4)

In § 2 Bolzano is presented as an adherent of the Aristotelian model of science. Section 3 discusses briefly Kant's view of the analytic-synthetic distinction; Bolzano studied it thoroughly. In § 4 his criticism of Kant's notion of analyticity is considered, while in § 5 and § 6 Bolzano's own characterization of this distinction is discussed. Section 7 connects Bolzano's notion of analyticity with his view of derivability or (logical) inference. In the following two sections this theme is further elaborated and developed in the light of the Aristotelian model of science and the notion of scientific demonstration implicit in it. Finally, § 10 presents some conclusions." (pp. 328-329)

(4) Cf. Y. Bar-Hillel, "Bolzano's Definition of Analytic Propositions." *Theoria* 16 (1950), pp. 91-117; p. 100. W. and M. Kneale, *The Development of Logic*, Oxford 1962, p. 367. J. Berg, "Introduction." In: B. Bolzano, *Theory of Science* (ed. by J. Berg; transl. by B. Terrell), Dordrecht 1973, pp. 12-44; p. 18. Coffa, *The Semantic Tradition*, Cambridge, 1991, p. 34.

22. ———. 2010. "The Analytic-Synthetic Distinction and the Classical Model of Science: Kant, Bolzano and Frege" *Synthese* no. 174:237-261  
Abstract: "This paper concentrates on some aspects of the history of the analytic-synthetic distinction from Kant to Bolzano and Frege. This history evinces considerable continuity but also some important discontinuities. The analytic-synthetic distinction has to be seen in the first place in relation to a science, i.e. an ordered system of cognition. Looking especially to the place and role of logic it will be argued that Kant, Bolzano and Frege each developed the analytic-synthetic distinction within the same conception of scientific rationality, that is, within the Classical Model of Science: scientific knowledge as *cognitio ex principiis*. But as we will see, the way the distinction between analytic and synthetic judgments or propositions functions within this model turns out to differ considerably between them."
23. Detlefsen, Michael. 2010. "Rigor, Re-proof and Bolzano's Critical Program." In *Construction. Festschrift for Gerhard Heinzmann*, edited by Bour, Pierre Edouard, Rebuschi, Manuel and Rollet, Laurent, 171-184. London: King's College Publications  
"Introduction  
The so-called critical movement in nineteenth and twentieth century foundational thinking(1) was described by the American mathematician George Miller (1863–1951) as one in which “[o]ur geometric intuitions are forced into the background” [27, p. 530] as, more and more, “logical deductions from definitions” (*loc. cit.*) take their place.  
The main sources of this movement, as both Miller and others described them, were the widely advertised problems concerning geometrical intuition as a guide to our thinking about continuity and differentiability. As mathematicians became increasingly sensitive to the press of these problems, they also “naturally became . . . more exacting in regard to rigor” (*loc. cit.*), and this renewed emphasis on rigor became the central element of nineteenth and early twentieth century attempts to “arithmetize” mathematics.  
How the notion of rigor mentioned was conceived and what its principal benefits were taken to be are prime concerns for me here. A better understanding of these matters should contribute to a better understanding of rigor and its motives and benefits overall. Therewith, I believe, should also come a fuller appreciation of the attention given to rigor by nineteenth century foundational thinkers. These at any rate are my chief goals here." (p. 171)  
(1) “Critical” was the term that was used by Felix Klein (cf. [20]) and various other writers (cf. e.g. [22]), F. Engel ([8]), J. Merz (cf. [25] and [26]), C. Keyser (cf. [17],

- [18] and [19]) and G. Kneebone (cf. [24]) to describe the proposals in the nineteenth century that called for the reformation of proof practices in mathematics, particularly analysis.
24. Drozdek, Adam. 1997. "Logic and Ontology in the Thought of Bolzano." *Logic and Logical Philosophy* no. 5:3-18  
 "Logic and theology were two domains of great importance to Bolzano. His attempt to reconcile the demands of these two domains led Bolzano to very strong logical realism, or, objectivism, whereby theology could be put on a firm ground. The paper analyzes the problem of objective concepts, propositions, and truths, with an attempt to give an interpretation of these entities, to account for their puzzling ontological status in Bolzano's system.  
 Bolzano is one of the forerunners of modern logic; however, his logical, and also mathematical, discussions were conducted in the context of very serious concern about the ontological status of the logical constructs. In the context of logic, he discusses the problem of propositions (*Sätze*) and their special category, namely truths; and ideas (*Vorstellungen*), and their special categories, namely intuitions (*Anschauungen*); and concepts. What is interesting in Bolzano's analyses is the considerable effort he devotes to distinguishing subjective propositions and ideas from objective propositions and ideas, the latter also called propositions and ideas in themselves. What is particularly puzzling in Bolzano's philosophy is the ontological status of the latter. According to Bolzano, objective propositions and ideas do not exist, they are not real, and yet they make logic possible." (pp. 3-4)
25. Dubucs, Jacques, and Lapointe, Sandra. 2006. "On Bolzano's Alleged Explicativism." *Synthese. An International Journal for Epistemology, Methodology and Philosophy of Science* no. 150:229-246  
 Abstract: "Bolzano was the first to establish an explicit distinction between the deductive methods that allow us to recognise the certainty of a given truth and those that provide its objective ground. His conception of the relation between what we, in this paper, call "subjective consequence", i.e., the relation from *epistemic reason* to consequence and "objective consequence", i.e., grounding (*Abfolge*) however allows for an interpretation according to which Bolzano advocates an "explicativist" conception of proof: proofs par excellence are those that reflect the objective order of grounding. In this paper, we expose the problems involved by such a conception and argue in favour of a more rigorous demarcation between the ontological and the epistemological concern in the elaboration of a theory of demonstration."
26. Duhn, Anita von. 2001. "Theoretical Laws and Normative Rules: Kant and Bolzano's Views on Logic." In *Kant und die Berliner Aufklärung. Akten des 9. Internationalen Kant-Kongresses. Band V: Sektionen XV-XVIII*, edited by Gerhardt, Volker, Horstmann, Rolf-Peter and Schumacher, Ralph, 3-12. Berlin: Walter de Gruyter  
 "Does logic instruct us how to think correctly? If so, what place does methodology have in logic? Is logic an instrument which provides rules for correct thinking or a system of proof for scientific theories, or is the doctrine of method merely an appendix to a doctrine of elements? The question whether logic is an *organon* is related to the question whether logical laws are theoretical truths or normative laws. Kant and Bolzano agree that logical laws basically provide us with truths, but that they can be apprehended as telling us how to think. (1) So a theoretical judgment that something is the case precedes the normative judgment that we may or should do something about it. Does it follow that Kant and Bolzano also agree on the question of whether logic is an *organon* which instructs us how to think? I will show that despite their divergent positions on logic, both authors claim that we apply normative rules because they are true." (p. 3)  
 (1) Kant and Bolzano agree with Husserl and Frege, who thought that a normative act, such as demanding or permitting, presupposes a theoretical act, such as judging or believing and that every law that states what *is* can be apprehended that one *ought* to think in accordance with it. Cf. Frege (1893) *Grundgesetze der Arithmetik*,

- intro. XV; Husserl (1900) *Prolegomena*, §§ 3, 13-14. I discuss this issue in "Is logic a theoretical or practical discipline? Kant and/or Bolzano", to appear in the *Archiv für Geschichte der Philosophie*. [vol. 84, no. 3 (2002) pp. 319-333]
27. ———. 2003. "Bolzano's Account of Justification." In *The Vienna Circle and Logical Empiricism: Re-evaluation and Future Perspectives*, edited by Stadler, Friedrich, 21-33. Dordrecht: Kluwer  
 "Bolzano investigated the following problem. How can we determine whether or not a certain truth is basic without recourse to subjective criteria based on intuition or immediate perceptual knowledge? For him, the criterion of self-evidence is not a means for justifying propositions because it does not provide us with a scientific proof presenting the objective reasons for a proposition, reasons that hold independently of our knowledge.(1) Bolzano intended to provide a workable alternative to the criterion of intuitive self-evidence, and claims that we have to search for proof even of self-evident propositions – at least until it becomes clear that and why no proof could be required.(2)  
 I reconstruct Bolzano's account of justification, which is designed to replace the criterion of self-evidence and provide a scientific basis for the demonstrative sciences. I then argue that although Bolzano succeeded in devising a procedure for grounding truths, his theory fails on the account that it implicitly reintroduces an epistemological problem." (p. 21)
- (1) 1804, § 3 (*Betrachtungen über einige Gegenstände der Elementargeometrie* (1804) in Bolzano's early mathematical works, Czechoslovak Studies in the History of Science, Prague, 1981. Partial English translation by S. Russ in W. Ewald, *From Kant to Hilbert*, vol. 1, OUP, 1996); *Beyträge II* (1810), §§2, 11, 12, 21 (*Beyträge zu einer begründeteren Darstellung der Mathematik* (1810) in Bolzano's early mathematical works, op.cit. (*Beyträge*). English translation by S. Russ in W. Ewald, *op.cit.*); 1817 (*Purely analytic proof...*), § 1; English translation by S. Russ in W. Ewald, *op.cit.*; WL IV, §525; (*Wissenschaftslehre* (1837), 4 vols, Aalen, Scientia Verlag, 1981 (WL)) and the *Anti-Euklid*, a manuscript in Bolzano's Nachlass edited by Karel Vecerka, Sbornik, Prague, 1967, pp. 204-215, who dates the text around 1840. Jan Sebestík, however, situates the text closer to 1816.  
 (2) 1804, §3.
28. Dummett, Michael. 1997. "Comments on Wolfgang Künne's Paper." *Grazer Philosophische Studien* no. 53:241-248  
 Comments on: W. Künne, *Propositions in Bolzano and Frege* (1997).  
*"Entertaining and judging (§§ 1,3)*  
 I feel some doubt about Wolfgang Künne's definition (E1)  
 x is a subjective idea  
 iff  
 x is not a judgement &  
 possibly for some y (y is a judgement and x is part of y).  
 A judgement must be some particular person's judgement, and occur at a particular time. Künne might now be judging that Bolzano misunderstood Kant; but suppose he is not. Are we to say that Künne now has a subjective idea of misunderstanding on the strength of the fact that he might be making that judgement?  
 The awkwardness arises from the difficulty of fitting (merely) entertaining or grasping a proposition into Künne's Figure 1. A proposition is indeed always something that it is possible to judge; but "*X entertains the proposition P*" cannot be defined as "*Possibly X judges that P*". Entertaining a proposition has to be acknowledged as a type of mental act in its own right, and as one more generic than judging: one that, like judging, has a proposition as its object (content, matter). Failure to acknowledge this leads to the complications of Figure 5." (p. 241)
29. Etchemendy, John. 1990. *The Concept of Logical Consequence*. Cambridge: Harvard University Press  
 Chapter 3: *Tarski on Logical Truth*, pp. 27-50.

"Though my concern in this book is not historical, a few preliminary words should be said about the complicated heritage of the model-theoretic definitions of the logical properties. As I mentioned, these definitions are generally credited to Tarski's 1936 article, and for the purposes of this book, there is no need to question this attribution.

What is clearly right about it is that Tarski's article contains the only serious attempt to state, in its most general form, the analysis underlying the standard definitions, and to put forward a detailed philosophical justification for that analysis. It is, so to speak, the philosophical locus of the model-theoretic definitions.

From a historical point of view, though, attributing the definitions to Tarski alone oversimplifies the situation a great deal.(4) For one thing, most of the main features of the analysis were anticipated, in various different ways, by earlier authors, including Bolzano (1837), Padoa (1901), Bernays (1922), Hilbert and Ackermann (1928), and Gödel (1929). Of all of these, Bolzano's discussion is by far the most extensive; in Chapter 3, I will briefly describe his account and motivate certain features of Tarski's analysis by comparing it with Bolzano's." (p. 7)

(4) For a more detailed discussion of the historical relationship between Tarski's analysis and the model-theoretic definitions, see Etchemendy (1988).

"I approach Tarski's account of logical truth and logical consequence indirectly, by considering first a simpler account developed by Bolzano nearly a century earlier. (1) The two accounts are remarkably similar; indeed, Tarski initially entertains what is, for all intents, precisely the same definition as Bolzano's, but modifies it for reasons I will eventually explain. But in spite of the striking similarity in the two accounts, Tarski was unaware of Bolzano's work until several years after the initial publication of his article. The key difference between the two accounts is simply that Bolzano employs substitution where Tarski uses the more technical, and for the purposes more adequate, notion of *satisfaction*." (p. 27).

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30. Fine, Kit. 2022. "Some Remarks on Bolzano on Ground." In *Bolzano's Philosophy of Grounding: Translations and Studies*, edited by Roski, Stefan and Schnieder, Benjamin. New York: Oxford University Press  
 "Bolzano takes the notion of grounding to be *factive*, so that grounds and consequences are always true propositions. But he also acknowledges that we sometimes invoke a non-factive notion of grounding.  
 Kit Fine discusses how Bolzano proposes to understand such a notion and critically assesses Bolzano's proposal from the perspective of the recent debate about grounding and its logic." (p. 37)

31. Føllesdal, Dagfin. 1981. "Comments on Quine." In *Philosophy and Grammar: Papers on the Occasion of the Quincentennial of Uppsala University*, edited by Kanger, Stig and Öhman, Sven, 29-35. Dordrecht: Reidel  
 Comments on W.V.O. Quine, *Grammar, Truth, and Logic*, same volume, pp. 17-28.  
 "I shall now comment on some points in the paper which, it seems to me, would be well worth discussing by our group. I will concentrate on the following three points:  
 1. Logical particles.  
 2. Syntactic ambiguities.  
 3. Demonstratives." (p. 29)  
 (...)
- "One hundred and fifty years ago, Bolzano was the first to have the idea of demarcating logic the way Quine does with the help of a set of logical particles which are held constant, while the other non-logical expressions are freely substituted for one another. However, Bolzano's idea received little attention until it was rediscovered afresh in the mid-thirties by Quine and Ajdukiewicz(3) independently of one another. All the basic ingredients are there in Bolzano: the steps that Bolzano goes through are the same as Quine's and in the same order:  
 1. Specify a vocabulary of logical particles.  
 2. Define what it means for two expressions to have the same logical form:  
 Two expressions have the same logical form if they can be obtained from one another by the substitution of non-logical expressions for non-logical expressions.  
 3. Define logical truth:  
 A sentence is logically true if and only if all sentences with the same logical form are true." (pp. 29-30, a note omitted)
- (3) Quine, W. V., 'Truth by Convention', in O. H. Lee (ed.), *Philosophical Essays for A. N. Whitehead* (Longmans, New York, 1936). Reprinted in W. V. Quine, *The Ways of Paradox* (Random House, New York, 1966), and in various other places, including Herbert Feigl and Wilfrid Sellars (eds.), *Reading in Philosophical Analysis* Appleton-Century-Crofts, New York, 1949). Ajdukiewicz, Kazimierz, 'Sprache und Sinn', *Erkenntnis* 4 (1934), 100-138.
32. ———. 1997. "Bolzano's Legacy." *Grazer Philosophische Studien* no. 53:1-11  
 Original German published as: *Bolzanos bleibende Leistungen* in: Arkadiusz Chrudzimski and Wolfgang Huemer (eds.), *Phenomenology and Analysis. Essays on Central European Philosophy*, Frankfurt: Ontos Verlag, 2004, pp. 57-68.  
 Abstract: "Bernard Bolzano (1781-1848) was an original and independent thinker, who left a lasting legacy in several areas of philosophy. Four such areas are singled for special attention: political philosophy, ethics and theology, logics and semantics, and mathematics. In all these areas he was far ahead of his time. He had pioneering ideas in political philosophy and in ethics and philosophy of religion, and he argued for them in a brilliantly clear way. In logic and semantics he anticipated Frege, Camap and Quine on important points, and he had intriguing, yet to be explored, ideas on intuition and other fundamental philosophical notions. In the foundations of mathematical analysis and the theory of infinite sets he anticipated Weierstrass and Cantor."
33. ———. 2001. "Bolzano, Frege and Husserl on Reference and Object." In *Future Pasts: The Analytic Tradition in Twentieth Century Philosophy*, edited by Floyd, Juliet and Shieh, Sanford, 67-80. Oxford: Oxford University Press  
 "Bolzano was a main influence on the development of Husserl's phenomenology. Husserl gives generous credit to Bolzano in several of his works and refers to him frequently. Husserl first came across Bolzano when, barely twenty, he read *Paradoxien des Unendlichen*(2) during his studies with Weierstrass in Berlin. And he renewed this acquaintance with *Paradoxien des Unendlichen* in 1884-1885 when he followed Brentano's lectures in Vienna on "Die elementare Logik und die in ihr notigen Reformen."  
 But it was only later, in the mid-1890s, that Husserl started serious study of Bolzano's *Theory of Science*,(3) which he earlier had regarded as "strange" ("fremdartig"). Husserl had then decided to give up work on the second volume of

- the psychologistic *Philosophy of Arithmetic* (1891) and had started working on what was to become his first phenomenological work, the *Logical Investigations* (1900-1901). Husserl states that he came to appreciate Bolzano, and in particular his theory of propositions (*Satze an sich*) and representations (*Vorstellungen an sich*), through studying Lotze's interpretation of Plato's theory of ideas. Husserl interpreted Bolzano in a platonistic manner, which Husserl claimed—I think unjustly—was foreign to Bolzano ([*Husseriana*] XXII, *Aufsätze und Rezensionen* (1890-1910)] p. 130)." (pp. 67-68)
- (2) Bernard Bolzano, *Paradoxien des Unendlichen*, ed. F. Prihonsky (Berlin: Mayer and Miiller, 1889; originally published 1851).
- (3) Bernard Bolzano, *Theory of Science*, abridged, ed. and trans. Rolf George (Berkeley: University of California Press, 1972, originally published 1837).
34. Fossati, Lorenzo. 2019. "Neither Aristotle nor Kant. Bernard Bolzano on Categories." In *Categories: Histories and Perspectives* 2, edited by D'Anna, Giuseppe and Fossati, Lorenzo, 77-94. Hildesheim: Georg Olms  
"The second Book of the *Wissenschaftslehre*, the *Elementarlehre* (Theory of Elements) is divided into four Parts; the first one is dedicated to the ideas in themselves (it is in the first volume of *Wissenschaftslehre*, the second volume take into account propositions in themselves, true propositions and inferences). This first Part, which includes §§ 46-120, is divided into four Chapters and introduces the notion of objective representation (often indicated as "idea"), its internal attributes and the distinction between the representations on the basis of their interrelation and of their relation to other objects (WL I: 214--571).  
Each paragraph is followed by some notes where Bolzano appeals to ancients and moderns to point out his own theses. At the end of the Chapter on ideas in themselves he adds a further Appendix (§§ 115-120) entitled "Previous Treatments of the Subject Matter of this Part," which helps better point out the big picture. In particular, two paragraphs are devoted to categories-§ 118 to the categories of the "ancients" and § 119 of the "moderns." He thus underlines his willingness to investigate any aspect and to involve all different kinds of interlocutors, but first and foremost his constant necessity to confront Aristotle and Kant." (pp. 77-78, notes omitted)
35. Fraisopi, Fausto. 2014. "The Quasi-Ontology of "An-Sich". Bernard Bolzano's *Theory of Science* between Leibnitian *Ars Combinatoria* and the Husserlian Idea of *mathesis universalis*." *Avello Publishing Journal* no. 4:1-25  
Abstract: "Starting from the critical position that Husserl assumes against Bolzano and his idea of *mathesis universalis*, this paper focuses and emphasizes Bolzano's project for a *mathesis* and the differences between this project and Leibniz's. Putting into an historical perspective these three forms of *mathesis*, by Leibniz, Bolzano, and Husserl, we / I open in so doing a theoretical perspective concerning the nonontological dimension of idealities they form and articulate *mathesis* as such. The an-ontological Combinatorics of propositions and of ideas in themselves, suggests, Bolzano maintains, the possibility of a treatment of Combinatorics independently from these ontological and metaphysical presuppositions that formed and structured the Leibnitian *ars combinatoria*. In this sense, the philosophical position of a "semantic Platonism," assumed by Bolzano, opens the perspective of a non-metaphysical but modular *mathesis* that we can articulate and widen beyond an ontological commitment."
36. Franks, Curtis. 2014. "Logical Completeness, Form, and Content: An Archaeology." In *Interpreting Gödel: Critical Essays*, edited by Kennedy, Juliette, 78-106. Cambridge: Cambridge University Press  
§ 2: *Bolzano's question*, pp. 81-92.  
"Bernard Bolzano engaged in the profound study of two distinct notions of logical consequence over several decades in the early nineteenth century.  
The work most remembered and highly regarded by modern logicians, because of its striking resemblance to twentieth century set-theoretical definitions of

consequence, concerns the Ableitbarkeit ("derivability") relation.

In his 1837 masterpiece, *Wissenschaftslehre*, Bolzano in fact defines a network of concepts – validity, compatibility, equivalence, and derivability – in terms of one another in a way very similar to contemporary presentations. Here is his definition of the last of these: [Wissenschaftslehre, § 155, text omitted]." (p. 81)

"Bolzano's two theories of logical consequence are themselves not precise enough for their correspondence with one another to be subject to proof. All the same, the question is at the center of Bolzano's thought.

The procedural *Ableitbarkeit* relation provides a calculus of inference.

The ontological *Abfolge* relation is a feature of the world absolutely independent of our ability to reason about it. By establishing that these notions correspond, we would ensure that the logical structure of the world is accessible, that some line of thought could trace the dependencies of truths, that the reasons behind the complex facts of reality are discoverable and comprehensible." (p. 92)

37. George, Rolf. 1961. *The Problems of the Infinite and the Continuum in Some Major Philosophical Systems of the Enlightenment*, Michigan State University Unpublished Ph.D thesis, available at Michigan State University, Digital repository. Contents: Introduction 1; Chapter I: Leibniz 19; Chapter II: Berkeley 63; Chapter III: Bayle 111; Chapter IV: Kant 133; Chapter V: Bolzano 192; Conclusion 215; Bibliography I-V.

"The philosophers discussed in this dissertation are Leibniz, Berkeley, Bayle, Kant, and Bolzano. Its aim is to show that certain difficulties connected with infinite and continuous sets were recognized by these philosophers, and that their systems were, at least in part, designed in such a way that these difficulties did not arise in them. (...)

Bolzano was the first to realize that the so-called Paradox of Galileo is no paradox at all, but simply describes a common property of all infinite sets.

As concerns the constitution of continua the problem was that neither the assumption that a continuum ultimately consists of unextended parts, nor that it consists of extended parts seemed defensible. Against the former case it was argued that unextended parts, no matter how many, cannot make a finite extension, against the latter that extended parts are not ultimate, but are further divisible. Bayle held that none of the logical alternatives are defensible, so that no one need bother to change whatever opinion he happens to have on the subject.

(...)

Bolzano declared that in a continuum every point has a neighbor within any distance, no matter how small. This definition, although ultimately unsatisfactory, proved to be of great help in discovering various important properties of continuous sets." (From the Abstract)

38. ———. 1972. "Enthymematic Consequence." *American Philosophical Quarterly* no. 9:113-116

"Enthymemes were traditionally defined as incomplete or incompletely stated syllogisms.

Arguments of this sort, though formally invalid, must be allowed to have some merit, and although the restriction to syllogisms is undesirable, the definition at least has the advantage of precision.

(...)

I shall argue in this paper that, while it is true that enthymematic arguments can be augmented so that valid arguments result, it is not wise to define enthymemes in these terms. I shall instead give a definition of enthymematic consequence which is similar to Tarski's definition of logical consequence; one can even arrange matters so that the latter becomes a limiting case of the former.

The definition can then be used to generate additional premisses which will convert enthymematic arguments into logically valid ones. It will thus automatically provide the desired restriction upon missing premisses.

I shall then show that the definition gives the same results as the traditional account within the domain of syllogisms, and that outside this domain it singles out a class

of invalid but plausible arguments which seem to answer to many logicians' intuition of what an enthymeme is, if we can take the examples in their textbooks as a clue." (p. 113)

(...)

"It remains to give a logician his due who more than a hundred years ago propounded a theory of logical consequence which in one definition accounted for both logical and enthymematic validity: Bernard Bolzano. He defined consequence thus: "I say that propositions  $M, N, O \dots$  follow from Propositions  $A, B, C, D, \dots$  with respect to the variable parts  $i, j, \dots$  if every class of representations whose substitution for  $i, j, \dots$  makes all of  $A, B, C, D, \dots$  true also makes all of  $M, N, O, \dots$  true."(6) It has been pointed out that Bolzano anticipated Tarski by almost exactly a hundred years in his definition of logical consequence.(7) Indeed, if the variable parts  $i, j, \dots$  are taken to consist of all and only the extralogical terms of  $A, B, C, D, \dots, M, N, O, \dots$  the definition is close to that of Tarski (though Bolzano demands that the premisses be consistent). Cases where  $i, j, \dots$  include more or fewer than the extralogical terms were generally regarded as somewhat quaint. In particular, it has not been seen that cases of Bolzano-entailment where the class of "variable" terms is smaller than the class of extralogical terms are just those argument forms which we are wont to call enthymemes." (p. 116)

(6) Bernard Bolzano, *Wissenschaftslehre*, Vol. 2 (Sulzbach, 1837), p. I 14.

(7) E.g., by Heinrich Scholz.

39. ———. 1983. "Bolzano's Consequence, Relevance and Enthymeme." *Journal of Philosophical Logic* no. 12:299-318

"Historians of logic tend to view their task as the application of modern insights and symbolic techniques to old texts. Perhaps they do this on the assumption that what is good in these works must be an adumbration of what was recently done and is now well known. This holds, at any rate, for most discussions of Bolzano's theory of logical consequence.

In the present paper I shall reverse this procedure and comment on some problems and beliefs of contemporary logic from what I take to be Bolzano's point of view. This will have the advantage of bringing out more forcefully than a straight exegesis what his view was and will also, I hope, put in doubt certain contemporary dogmas. I begin by applying his definition of consequence to propositional logic. Bolzano did not entertain this branch of logic, and to this extent my account is ahistorical. That it is, nonetheless, a straight extension of his theory is shown by the fact that all 23 theorems about consequence which he proves in his *Theory of Science* hold in this application I then consider how C. I. Lewis's so-called "independent proof" for  $A \& \neg A \models B$  fares in this system (it fails). After some comments on the proof, I show that in Bolzano-consequence premisses and conclusion share a subsentence (a necessary condition of relevance). There follows a discussion of enthymemes and a general procedure for generating the so-called "nutting premiss". At the end I sketch a taxonomy of consequence relations and briefly remark on earlier interpretations of Bolzano's work. In using the first person plural (from now on) I mean to speak for those who think Bolzano's approach sound, a group that includes at least Bolzano and myself." (p. 299, notes omitted)

40. ———. 1983. "A Postscript on Fallacies." *Journal of Philosophical Logic* no. 12:319-325

"Bolzano is justly esteemed for his opposition to psychologism in logic. It is most fitting, therefore, that his definition of consequence has enabled us to strike a blow at the residual psychologism that is found in the customary treatment on enthymemes.(1) We shall now do the same for the so-called formal fallacies." (p. 319)

(1) See section (9) of the preceding essay. [*Bolzano's Consequence, Relevance and Enthymeme*]

41. ———. 1986. "Bolzano's Concept of Consequence." *Journal of Philosophy* no. 83:558-564

Reprinted in: Dale Jacquette (ed.), *Philosophy of Logic: An Anthology*, Malden: Blackwell, 2002, pp. 205-209.

"Plainly, to identify a speech as an argument and to understand its premises and conclusion is not the same as knowing what argument is intended. What is missing? Bernard Bolzano defines the concept of consequence thus:

Propositions *M, N, O, . . .* follow from propositions *A, B, C, D, . . .* with respect to variable parts *i, j, . . .* if every class of ideas whose substitution for *i, j, . . .* makes each of *A, B, C, D, . . .* true also makes all of *M, N, O, . . .* true.(1)

The *i, j, . . .* are constants tagged for substitution; I shall call them *variands*." (p. 558)  
(...)

"The conception of consequence here adumbrated has two features that should recommend it to logicians who are concerned not with the development of formal systems, but with the analysis of informally stated arguments and the identification of fallacies. The first of these is that arguments of invalid form are invalid. In the classical view, this is not the case, as Gerald Massey has pointed out with clarity and vigor.(3)

(...)

In Bolzano's view, the evaluation of any argument must begin with the identification of variands. If their variation generates an invalid form, the argument is invalid; if not, not. It is of course possible to make mistakes in this, just as sentences can be misunderstood. It is a cultural, and perhaps even a human, failing that we do not usually indicate the variands explicitly. But these are problems of communication. Plainly, it is often possible, and sometimes important, to identify formal fallacies. It therefore seems that in this respect Bolzano's account of consequence is superior to the classical. A second positive feature of Bolzano's conception is that it gives a promising account of enthymemes. Although he concentrates on arguments in which all indexical elements are variands (this being the proper province of logic, cf. WL § 223), his definition does not exclude cases in which only some of them are. We readily identify 'Socrates' as the variand in 'Socrates was a man, therefore Socrates was mortal'. That is, we understand this argument as implicitly claiming that every substitution on 'Socrates' that makes the premise true also makes the conclusion true. If we had to construct a device for computing the "missing premise" (which we intuitively take to be 'All men are mortal'), we would have it state that fact. It would, that is, form the universal closure on the variand, over the conditional consisting of premise and conclusion, and voila, the missing premise results. This procedure works for all syllogistic enthymemes, and is only slightly more complex when no singular terms are involved. No principle of charity or other proviso is needed. I venture the guess that some such computation is going on even in our own minds when, with a speed that must compel wonder, we determine what all the world takes to be the missing premise in such a case."

(1) *Wissenschaftslehre* (Sulzbach, 1837), § 155, no. 2, vol. ii, pp. 199 ff. Translated as *Theory of Science*, R. George, ed. (Oxford: Blackwell, 1972), p. 209. Henceforth WL.

(3) "The Fallacy behind Fallacies," in P. A. French, T. E. Vehling, Jr., and H. K. Wettstein, eds., *The Foundations of Analytic Philosophy* (Minneapolis: Minnesota UP, 1981), pp. 499 ff.

42.

———. 1987. "Bolzano on Time." *Philosophia Naturalis* no. 24:452-468

"In the first volume of the *Wissenschaftslehre* Bolzano claims that "by the word 'time' we mean nothing but that particular determination in a real thing which is the condition for correctly attributing to it a given property."(1) He says that from this *all* properties of time can be deduced. This is supported by just one example, namely, that several contrary properties can be attributed to the same substance only on condition that times differ. This follows directly, since sentences with contrary predicates can be true only if their subjects differ. Hence one and the same substance can have contrary attributes only on the assumption that its time determinations are not the same.

In Chapter 412 he maintains that a theory can have the status of a science even if its extent is very small. Consequently, he says, "the theory of time (the properties of time, not of the art of measuring it) deserves to be treated as a special science (i.e. the pure theory of time) although this science can consist of only a very few propositions."(2) Kant, he objects, should not have denied it the name of science for no other reason than its small extent.

In the following chapter Bolzano adds that a theory need not be denied the status of a science even if everyone already knows its propositions. Again the theory of time serves as an example. He maintains that all theorems of the pure theory of time are obvious to everyone (*sind jedem von selbst schon bekannt*) (3), but that it should be considered to be a science nonetheless.

These are sweeping claims. Given the voluminous publications, the many controversies and the continuing interest in the subject of time they seem strange, even absurd. I begin by discussing these assertions, then add some reflexions on Bolzano on *time perception*, and end with a brief account of his criticism of Kant's views." (p. 452)

- (1) *Wissenschaftslehre* I, 365. Citations follow the first edition.
- (2) IV, 52.
- (3) IV, 53.

43. ———. 1992. "Concepts of Consequence." In *Bolzano's Wissenschaftslehre 1837-1987. International Workshop*, 3-26. Firenze: Leo S. Olschki

"It has been held since antiquity that in all deductive argumentation there is a formal element or aspect. I wish to distinguish, and contrast, two ways of characterizing this. One of them I call «logic of schemata», or the «Received View», and the other, which was first articulated by Bolzano, «logic of variation». I shall investigate how these concepts of consequence succeed in addressing five concerns, not all of them logical issues, as we now understand logic, but connected with argumentative practice and certain epistemic matters.

(1) For the sake of completeness I mention first that a definition of consequence should fix a relation that satisfies certain formal requirements, i.e. a cut rule, thinning, and the like. There is a conventionally accepted set of these, described, e.g. by Gentzen. If a consequence relation shows deviations from this, it must be a reasoned difference that should be argued for. Also, a consequence relation (specifically logical, rather than enthymematic consequence) should be defined in such a way that first order predicate logic is strongly complete, that is, that if A is a consequence of a set of sentences X, then A should be deducible from X in a finite sequence of steps.

(2) A defensible definition of consequence should have the form, broadly, «If an argument satisfies this definition, it is valid, otherwise not». Contemporary definitions fail, as a rule, to satisfy the «otherwise not» clause. It is, however, argumentative practice to convict arguments of being formally fallacious. This can only be based on the assumption that if we have *fully understood* an argument, we can judge it to be valid or invalid - setting aside such esoterica as undecidable cases. I think it desirable that a definition of consequence allow an account of invalidity as well as validity.

(3) I shall consider a definition of logical consequence to be superior if it is broad enough to explain why we concede merit to some formally invalid arguments (enthymemes), but withhold approbation from others (gross non-sequiturs), that is, if it treats *logical consequence* as a special, though perhaps the most important and interesting, case.

(4) Arguments as presented in both informal and formal contexts can be ambiguous, even if they are constructed of unambiguous sentences, and even if they are couched in a language that stipulates a rigid distinction between logical and extralogical constants. I call an argument *naked* if all that is presented are premisses, conclusion, and an inference indicator, like «therefore». I shall maintain that when we understand an argument, we understand more than the sentences of which it is composed, and more than the unspecified claim that the conclusion *somehow*

follows from the premisses. That is, we grasp more than the naked argument. If we fail in this, we may misconstrue arguments, which amounts to saying that naked arguments can be ambiguous. I suggest that an acceptable theory of consequence should allow us to bring into focus the problem of argument ambiguity.

(5) It is desirable that a concept of consequence, if it does not itself define a «relevant» relation, can at least be augmented so that it does. (A consequence relation is here called relevant if it stipulates or implies that premisses and conclusion share some element)." (pp. 3-4)

44. ———. 1997. "Psychologism in Logic: Bacon to Bolzano." *Philosophy and Rethoric* no. 30 (3):213-242

Reprinted in: Dale Jacquette (ed.), *Philosophy, Psychology, and Psychologism. Critical and Historical Readings on the Psychological Turn in Philosophy*, Dordrecht: Kluwer, 2003, pp. 21-49.

"The first logician to conceive of logic as a matter wholly apart from psychology was Bolzano. He did, however not neglect the old concerns. Of the four volumes of his *Wissenschaftslehre* only the first two (and not all of them) deal with the objective world of propositions in themselves. The third is epistemology, dealing with the manifestation of propositions in the mind: a judgment, in contrast to a proposition, which is abstract and mind independent, now is a mental episode whose "matter" is a proposition in itself. In this part of the work he discusses all those issues that tended to be mixed into the discussion of logic itself: clarity and obscurity of representations, knowledge and error, as well as the "art of discovery" which now has its proper place as a part of epistemology. The last volume, finally, is given over to the presentation of a science in the form of a treatise of the subject. This is the old "methodology", the theory of combining discovered truths into the system of a science." (p. 39 of the reprint)

(...)

"I hope to have clarified in this paper at least some of the strands of psychologism that ran through the history of logic between Bacon and Bolzano. Much had to be left out. My thesis — if I may be said to have argued one — has been that there were different kinds of intrusion of psychology into logic, some due to a conception of logic that included much of what is now assigned to other fields, others due to cultural and ideological persuasions, and still others to the obsession that logic is the science of thinking." (p. 44 of the reprint)

45. ———. 1997. "Bolzano's Programme and Abstract Objects." *Grazer Philosophische Studien* no. 53:167-180

Abstract: "Most of the Bolzano literature is exegetical, neglecting, unfortunately, the great potential of his logic as the beginning of a *Programme*. Specifically, his unorthodox construal of the consequence relation as triadic, and his account of logical form are promising beginnings which even as they stand shed light on question of relevance, the ancient problems of enthymemes and others. Instead of developing these suggestions, Bolzano scholars have been occupied with elucidating the ontology of sentences in themselves, and related topics. I argue, and believe to be in agreement with Bolzano, that the nature of sentences is fully explained by the relations that hold between them, just as money has no nature or essence beyond the transactions it makes possible. It follows that the development of his logic would contribute at least as much to the understanding of sentences than any exegesis."

46. ———. 2003. "Bolzano and the Problem of Psychologism." In *Husserl's Logical Investigations Reconsidered*, edited by Fisette, Denis, 95-108. Dordrecht: Kluwer  
"As we saw, the view that subjective ideas are parts of judgments was not new, but Bolzano's theory of objective contents allowed him to avoid a certain confusion. It was generally acknowledged that ideas pass through the mind when one thinks, i.e. judges. At the same time they were thought to be sensations, or the copies of sensations, that is, visual or auditory sense data.

They were often described in terms not consistent with their roles as terms of judgments, i.e. as extended, round, moving, receding, as semblances of their objects, etc. (cf. Exner, *supra*) [\*]. But mental occurrences of this sort cannot be terms of judgments. Hume, for example, claimed reasoning to be the operation of our thoughts and ideas, but it is not very plausible to think of it as an operation on something that can be blue, round, divided, or point-like.

Bolzano was not a victim of that confusion. For him a subjective idea is part of a mental proposition or at least could be such a part, and must have the character that goes with this role. He concentrates on the logical functions of ideas, thus avoiding certain classical mistakes. In particular the view that knowledge consists in the similarity or resemblance between our ideas and their objects is exposed as fallacious. Terms of propositions refer to their objects, they need not resemble them. The truth of a proposition, and hence our knowledge of an object, does not depend upon the similarity between idea and object. Rather, "a proposition is true if we connect with the idea of an object the idea of an attribute which this object actually has" (WL §42). This rejection of the resemblance theory is not based on the classical argument that we can never know whether our ideas resemble their objects since we can never compare the two, the object being altogether inaccessible. Rather, the critical point is that it is of no consequence whether an idea resembles its object." (pp. 105-106)

(...)

"Bolzano had a very generous conception of the scope of logic, which for him included a logic of discovery, epistemology and a lot of communication theory. He insisted that logic in this broad sense needed to make use of psychological theory. However, the Theory of Elements in the first two volumes of WL on which "logic as a science must be built" (Husserl) is a historical first in avoiding all connection with psychological doctrine." (p. 108)

[\*] Bernard Bolzano, *Letter to Franz Exner*, 18th December 1834, in: *On the Mathematical Method and Correspondence with Exner*, Amsterrdfam: Rodopi 2004, pp.157-174.

47. ———. 2004. "Intuitions—the Theories of Kant and Bolzano." In *Semantik und Ontologie. Beiträge zur philosophischen Forschung*, edited by Siebel, Mark and Textor, Mark, 319-354. Frankfurt: Ontos Verlag  
 "Bolzano credits Kant with impressing on the philosophical public the distinction between intuition (*Anschauung*) and concept (*Begriff*). But making the distinction is one thing, explaining it is another. Bolzano is not happy with Kant's account (WL I, § 77),(1) but his critique does not connect well with Kant's theory. The gulf between them, in both substance and terminology, is too deep. Despite the divergence between the two philosophers on almost any topic, Bolzano paid more attention by far to Kant and Kantian logicians than any other tradition or school, for good reasons."

(...)

"Mathematical propositions are purely conceptual, and so intuitions will play no role in their proof or analysis. They can be established a priori because they are purely conceptual. Bolzano's theory of intuition supports this profoundly important tenet of his thought. His redefinition of "Anschauung" was thus not merely an exercise in persuasive definition, and the appropriation of a popular and important expression for different purposes.

According to Bolzano (and in truth, I might add) there are no such things as Kantian intuitions. Bolzano's construal of the word, whatever its shortcomings, certainly removes the temptation to seek geometrical and arithmetic truth in intuitions, yet preserves the root connotation that *Anschauungen* are those thought episodes that represent our direct empirical awareness." (p. 35)

(1) Bolzano 1837. The *Wissenschaftslehre* is cited as *WL* plus number of volume.